Ministero dell'Istruzione, dell'Università, della Ricerca
Ufficio Scolastico Regionale Per Il Lazio
Liceo Scientifico Statale
"TALETE"

Modulo 3 MiniCLIL

Incontro 9: Sequences and series/2

L'incontro PON di oggi si propone di continuare la riflessione sulla serie di numeri e sequenze, e di introdurre a livello applicativo il tema del prossimo incontro che ci porterà a conoscere la figura del genio matematico indiano Srinivasa Ramanujan che ha illuminato il modo matematico nell'ambito della teoria analitica dei numeri e si è rivelato artefice di formule sommatorie capaci di aver rivoluzionato il campo matematico-scientifico futuro.

Lezione

L'UdA si articolerà sulla lettura e discussione di un testo esplicativo di una sommatoria non convergente e sul problema non commutatività della somma nel caso infinito. Questa serie viene ricondotta ad uno dei lavori del matematico britannico. La trattazione e il confronto tra pari ha lo scopo di utilizzare la terminologia necessaria in lingua inglese per l'esplicazione dei processi di calcolo intuiti da Ramanujan, e servirà al contempo come introduzione al film biografico del grande matematico indiano che verrà proiettato e discusso nella lezione successiva.

In allegato il materiale fornito a lezione.

The Ramanujan Summation: 1 + 2 + 3 + ⋯ + ∞ = -1/12?

“What on earth are you talking about? There’s no way that’s true!” — My mom

This is what my mom said to me when I told her about this little mathematical anomaly. And it is just that, an anomaly. After all, it defies basic logic. How could adding positive numbers equal not only a negative, but a negative fraction? What the frac?

Before I begin: It has been pointed out to me that when I talk about sum’s in this article, it is not in the traditional sense of the word. This is because all the series I deal with naturally do not tend to a specific number, so we talk about a different type of sums, namely Cesàro Summations. For anyone interested in the mathematics, Cesàro summations assign values to some infinite sums that do not converge in the usual sense. “The Cesàro sum is defined as the limit, as n tends to infinity, of the sequence of arithmetic means of the first n partial sums of the series” — Wikipedia. I also want to say that throughout this article I deal with the concept of countable infinity, a different type of infinity that deals with a infinite set of numbers, but one where if given enough time you could count to any number in the set. It allows me to use some of the regular properties of mathematics like commutativity in my equations (which is an axiom I use throughout the article).

For those of you who are unfamiliar with this series, which has come to be known as the Ramanujan Summation after a famous Indian mathematician named Srinivasa Ramanujan, it states that if you add all the natural numbers, that is 1, 2, 3, 4, and so on, all the way to infinity, you will find that it is equal to -1/12. Yup, -0.08333333333.

Don’t believe me? Keep reading to find out how I prove this, by proving two equally crazy claims:

1–1+1–1+1–1 ⋯ = 1/2

1–2+3–4+5–6⋯ = 1/4

First off, the bread and butter. This is where the real magic happens, in fact the other two proofs aren’t possible without this.

I start with a series, A, which is equal to 1–1+1–1+1–1 repeated an infinite number of times. I’ll write it as such:

A = 1–1+1–1+1–1⋯

Then I do a neat little trick. I take away A from 1

1-A=1-(1–1+1–1+1–1⋯)

So far so good? Now here is where the wizardry happens. If I simplify the right side of the equation, I get something very peculiar:

$1\text{-}A=1-1+1-1+1-1+1\cdots$

Look familiar? In case you missed it, thats A. Yes, there on that right side of the equation, is the series we started off with. So I can substitute A for that right side, do a bit of high school algebra and boom!

$1\text{-}A = A$

$1\text{-}A+A=A+A$

$1 = 2A$

$1/2 = A$

This little beauty is Grandi's series, called such after the Italian mathematician, philosopher, and priest Guido Grandi. That's really everything this series has, and while it is my personal favourite, there isn't a cool history or discovery story behind this. However, it does open the door to proving a lot of interesting things, including a very important equation for quantum mechanics and even string theory. But more on that later. For now, we move onto proving #2: $1-2+3-4+5-6\cdots = 1/4$.

We start the same way as above, letting the series $B = 1-2+3-4+5-6\cdots$. Then we can start to play around with it. This time, instead of subtracting B from 1, we are going to subtract it from A. Mathematically, we get this:

$A\text{-}B = (1-1+1-1+1-1\cdots) - (1-2+3-4+5-6\cdots)$

$A\text{-}B = (1-1+1-1+1-1\cdots) - 1+2-3+4-5+6\cdots$

Then we shuffle the terms around a little bit, and we see another interesting pattern emerge.

$A\text{-}B = (1-1) + (-1+2) + (1-3) + (-1+4) + (1-5) + (-1+6)\cdots$

$A\text{-}B = 0+1-2+3-4+5\cdots$

Once again, we get the series we started off with, and from before, we know that A = 1/2, so we use some more basic algebra and prove our second mind blowing fact of today.

$A\text{-}B = B$

$A = 2B$

$1/2 = 2B$

$1/4 = B$

And voila! This equation does not have a fancy name, since it has proven by many mathematicians over the years while simultaneously being labeled a paradoxical equation. Nevertheless, it sparked a debate amongst academics at the time, and even helped extend Euler's research in the Basel Problem and lead towards important mathematical functions like the Reimann Zeta function.

Now for the icing on the cake, the one you've been waiting for, the big cheese. Once again we start by letting the series C = 1+2+3+4+5+6⋯, and you may have been able to guess it, we are going to subtract C from B.

B-C = (1–2+3–4+5–6⋯)-(1+2+3+4+5+6⋯)
Because math is still awesome, we are going to rearrange the order of some of the numbers in here so we get something that looks familiar, but probably wont be what you are suspecting.

B-C = (1-2+3-4+5-6⋯)-1-2-3-4-5-6⋯

B-C = (1-1) + (-2-2) + (3-3) + (-4-4) + (5-5) + (-6-6) ⋯

B-C = 0-4+0-8+0-12⋯

B-C = -4-8-12⋯

Not what you were expecting right? Well hold on to your socks, because I have one last trick up my sleeve that is going to make it all worth it. If you notice, all the terms on the right side are multiples of -4, so we can pull out that constant factor, and lo n' behold, we get what we started with.

B-C = -4(1+2+3)⋯
B-C = -4C
B = -3C
And since we have a value for B=1/4, we simply put that value in and we get our magical result:

1/4 = -3C
1/-12 = C or C = -1/12
Now, why this is important. Well for starters, it is used in string theory. Not the Stephen Hawking version unfortunately, but actually in the original version of string theory (called Bosonic String Theory). Now unfortunately Bosonic string theory has been somewhat outmoded by the current area of interest, called supersymmetric string theory, but the original theory still has its uses in understanding superstrings, which are integral parts of the aforementioned updated string theory.

The Ramanujan Summation also has had a big impact in the area of general physics, specifically in the solution to the phenomenon know as the Casimir Effect. Hendrik Casimir predicted that given two uncharged conductive plates placed in a vacuum, there exists an attractive force between these plates due to the presence of virtual particles bread by quantum fluctuations. In Casimir's solution, he uses the very sum we just proved to model the amount of energy between the plates. And there is the reason why this value is so important.

So there you have it, the Ramanujan summation, that was discovered in the early 1900's, which is still making an impact almost 100 years on in many different branches of physics, and can still win a bet against people who are none the wiser.

P.S. If you are still interested and want to read more, here is a conversation with two physicists trying to explain this crazy equation and their views on it's usefulness and validity. It's nice and short, and very interesting. https://physicstoday.scitation.org/do/10.1063/PT.5.8029/full/